Math 249 Lecture 35 Notes

Daniel Raban

November 15, 2017

1 Dimension of Covariant Rings and Chevalley's Theorem

1.1 Dimension of covariant rings

Last time we talked about invariant rings. Let $G \circlearrowright K^n$, where K is a field with characteristic 0, and let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables. Then R^G is the ring of invariant polynomials. We also have $I_G = (R_+^G)$, the ideal generated by the invariants with 0 constant term. We can form the coinvariant ring R/I_G, in which the only invariant elements are constants. We also proved the following lemma:

Lemma 1.1. *The homogenous $f_i \in R_+^G$ generate R^G as a K-algebra iff they generate I_G as an ideal.*

Proposition 1.1. *Assuming the action $G \circlearrowright K^n$ is faithful,*

$$\dim_K R/I_G \geq |G|.$$

Proof. We claim that homogenous basis X of R/I_G (picking the representatives for the cosets) will generate R as an R^G-module. We want to show that $R^G \cdot X = R$. Let $f \in R$ be $f = \sum c_i a_i + \sum \varphi_j g_j$, where $g_j \in R_+^G$. By induction, $\varphi_j \in R^G \cdot X$.

Find a $v \in K^n$ such that $|G \cdot v| = |G|$. R needs at least $|G|$ generators to generate even this orbit. $\square$

Example 1.1. Let $G = \{\pm 1\} \circlearrowright K^2$ by scalar multiplication by ± 1, and let $R = K[x, y]$. R^G is the set of even polynomials, $K[x^2, xy, y^2]$. Also, $I_G = (x^2, xy, y^2)$. R/I_G has a basis $\{1, x, y\}$. So $\dim(R/I_G) = 3 > 2 = |G|$.

This is not a Coxeter group! Abstractly, the group is, but the action is not the Coxeter group action.

Example 1.2. Let $S_n \circlearrowright \mathbb{R}^n$. $R = \mathbb{R}[x_1, \dots, x_n]$, $\mathbb{R}^{S_n} = \mathbb{R}[e_1, \dots, e_n]$, and $I_G = (e_1, \dots, e_n)$. I_{S_n} is a *complete intersection ideal*, so $\dim(R/I_{S_n}) = \prod \deg(e_i)$ (although we will not digress to prove this here. So $\dim(R/I_{S_n}) = n! = |S_n|$. In fact, R is a free module of rank $n! = |S_n|$ over R^{S_n}.

1.2 Chevalley's theorem and consequences

Theorem 1.1 (Chevalley). *If $G \circlearrowright \mathbb{R}^n$ is a Coxeter group, then*

1. *R^G is a polynomial ring.*
2. *I_G is a complete intersection ideal.*
3. *R is a free R^G-module (of rank $|G|$) with basis any R-bases of R/I_G.*
4. $\dim R/I_G = |G|$.

Before we prove this, let's go over some consequences. Define the generating function

$$H_{R/I_G}(q) := \sum_d \dim(R/I_G)q^d.$$

We also have that

$$H_R(q) = \frac{1}{(1-q)^n},$$

$$H_{R^G}(q) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)},$$

since $1, 2, \ldots, n$ are the degrees d_i or $e_1, \ldots, e_n$.

So $H_R = H_{R^G} H_{R/I_G}$ because

$$H_{R/I_G}(q) = \frac{1-q}{1_q}\frac{1-q^2}{1-q}\cdots\frac{1-q^n}{1-q} = [1]_q[2_q]\cdots[n]_q = [n]_q!$$

In S_n, $\ell(w)$ is the number of inversions. So we can say

$$H_{R/I_{S_n}}(q) = \sum_{w \in S_n} q^{\ell(w)}.$$

When $S_n \circlearrowright \mathbb{R}^n$, this is a degenerate action. Let $R = \mathbb{R}[u_1, \ldots, u_{n-1}] = R/(e_1)$. Let $\tilde{R}^{S_n} = R^{S_n} = \mathbb{R}[e_2, \ldots, e_n]$. Then $\tilde{R}/\tilde{I}_G \cong R/I_G$. So e_1 is the generator for the degenerate part.

$S_2 \circlearrowright \mathbb{R}$ is actually the action of $\{\pm 1\}$. So our example before was $S_n \circlearrowright \mathbb{R} \oplus \mathbb{R}$. Let $S_n \circlearrowright \mathbb{R}^{n-1}$ (or $\mathbb{R}^n$), and call this V. Then $S_n \circlearrowright V \oplus V$ and, remarkably, $\dim(R/I_{S_n} = (n+1)^{n-1}$. This is a deep fact! This is the number of rooted trees on $n+1$ vertices. There is no known simple proof of this.